# Quick energy drop in Stochastic 2D Minority

Damien Regnault[1]

Université de Lyon, IXXI-LIP, École Normale Supérieure de Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France. http://perso.ens-lyon.fr/damien.regnault

**Abstract.** Cellular automata are usually updated synchronously and thus deterministically. The question of stochastic dynamics arises in the development of cellular automata resistant to noise [1] and in simulation of real life systems [2]. Synchronous updates may not be a valid hypothesis for such simulations and most of these studies use stochastic versions of cellular automata.
In [3–6], the authors study different classes of cellular automata under fully asynchronous dynamics (only one random cell fires at each time step) and $\alpha$-asynchronous dynamics (each cell has a probability $\alpha$ to fire at each time step). They develop tools and methods to ease the study of other cellular automata. In [4, 6], they analyze 2D Minority under fully asynchronous dynamics for Von Neumann and Moore neighborhoods. The behavior of this cellular automaton under these dynamics is surprisingly rich. The energy of a configuration is an useful information. In [4], it is proved that configurations of energy greater than $\frac{5mn}{3}$ (where $m$ and $n$ are the length and the width of the configuration) will not appear in the long range behavior of 2D minority for Von Neumann neighborhood. In this paper we improve this bound to $18\lceil\frac{m}{4}\rceil\lceil\frac{n}{4}\rceil$. The proof is based on an enumeration of cases made by computer. This method could be easily tuned for other cellular automata or neighborhoods.

## 1 Introduction

Cellular automata are attractive models for complex systems in various fields, like physics, biology or social sciences. An example of challenging issue in biology is to predict the expression of genes in a set of cells which share the same gene regulatory network. Cellular automata can be used to model such systems [2, 7]. Classically cellular automata update synchronously. However models for natural phenomena rather update asynchronously.

Empirical studies [8–10] have shown how widely the behavior can change when introducing asynchronism. However only few mathematical analyses are available and they mainly concern one-dimensional stochastic cellular automata [3, 5, 11]. Providing analyses of 2D rules remains a real challenge. For instance the mean-field approach does not succeed in approximating tightly such stochastic dynamics [12]. The cellular automaton 2D Minority is studied under fully asynchronous dynamics (at each time step only one random cell chosen uniformly fires) in [4, 6]. 2D Minority is a rule with negative feedback. Such rules are acknowledged to be harder to analyze [13]. In [4, 6] the authors develop tools for

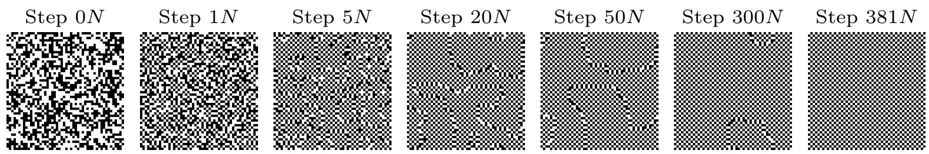


**Fig. 1.** A typical evolution of 2D Minority under fully asynchronous dynamics on a configuration of $N = 50 \times 50$ cells.

studying 2D asynchronous cellular automata. Ongoing works show their results hold for the classes of 2D Threshold cellular automata. This class has been intensively studied under synchronous dynamics [14] and exhibits interesting behaviors under fully asynchronous dynamics.

In this paper, we continue the study of 2D Minority under fully asynchronous dynamics with Von Neumann neighborhood. Figure 1 shows a classical evolution of fully asynchronous 2D Minority. In [4], it is proved that any initial configuration will reach almost surely a stable configuration after $O(N^{2N+1})$ steps on expectation (where $N$ is the number of cells of the configuration). Nevertheless it is conjectured that this time is polynomial in $N$. The energy of a configuration is useful to describe the behavior of 2D Minority. This notion was first introduced in Ising model [15] or Hopfield networks [13]. In [4], it is shown that:

1. The energy of a configuration is between 0 and $4N$.
2. The energy of a stable configuration is between 0 and $N$.
3. The energy is non-increasing according to time.
4. The energy drops below $5N/3$ after $O(N^2)$ steps on expectations.
5. The dynamics converges almost surely to a stable configuration from an initial *bounded* configuration in $O(N^2)$ steps on expectation. Bounded configurations correspond to final steps of a classical execution of $2D$ Minority (After step $50N$ the configurations of figure 1 are bounded configurations).

In this paper we improve point 4: the energy drops below $18\lceil \frac{m}{4} \rceil \lceil \frac{n}{4} \rceil$ after $O(N^2)$ steps on expectation (where $m$ and $n$ are the length and the width of the configuration: $N = mn$). The proof relies on an enumeration of cases by computer. Nevertheless, our method could be easily adapted for other cellular automata or neighborhoods. Moreover, a more accurate bound could be found with more precise computations (but cannot be lower than $N$ because of point 2). Finally our results is of interest to prove the conjecture. Only a small gap separates configurations of energy less than $18\lceil \frac{m}{4} \rceil \lceil \frac{n}{4} \rceil$ and bounded configurations.

## 2 Definitions

We consider in this paper the 2D 2-states cellular automaton Minority under fully asynchronous dynamics over finite configurations with periodic boundary conditions. Except for the main theorem, all notations and results are introduced or proved in [4].

**Definition 1 (Configuration).** *We are given two positive integers $n$ and $m$, let $N = nm$. We denote by $\mathbb{T} = \mathbb{Z}_n \times \mathbb{Z}_m$ the set of* cells *and $Q = \{0, 1\}$ the set of* states *(0 stands for white and 1 for black in the figures). We consider the* Von Neumann neighborhood*: the neighbors of each cell $(i, j)$ are the four cells $(i, j \pm 1)$ and $(i \pm 1, j)$. A $n \times m$*-configuration *$c$ is a function $c : \mathbb{T} \to Q$; $c_{ij}$ is the* state *of the cell $(i, j)$ in configuration $c$.*

**Definition 2 (Stochastic 2D Minority).** *We consider the* fully asynchronous dynamics of 2D Minority*. Time is discrete and let $c^t$ denote the configuration at time $t$; $c^0$ is the* initial configuration*. The configuration at time $t+1$ is a random variable defined by the following process: a cell $(i, j)$ is selected uniformly at random in $\mathbb{T}$ and its state is updated to the minority state in its neighborhood (we say that cell $(i, j)$* fires *at time $t$), all the other cells remain in their current state:*

$$c_{ij}^{t+1} = \begin{cases} 1 \text{ if } (c_{ij}^t + c_{i-1,j}^t + c_{i+1,j}^t + c_{i,j-1}^t + c_{i,j+1}^t) \leq 2 \\ 0 \text{ otherwise} \end{cases}$$

*and $c_{kl}^{t+1} = c_{kl}^t$ for all $(k, l) \neq (i, j)$. A cell is said* active *if its state would change if fired.*

**Definition 3 (Potential).** *The* potential *$v_{ij}$ of cell $(i, j)$ is the number of its neighboring cells in the same state as itself. By definition, if $v_{ij} \leq 1$, then the cell is in the minority state in its own neighborhood and is thus* inactive *(its state will not change when fired); whereas, if $v_{ij} \geq 2$ then the cell is* active *and its state will change if fired.*

**Definition 4 (Energy).** *The* energy *$E$ of configuration $c$ is defined as: $E = \sum_{(i,j)\in\mathbb{T}} v_{ij}$.*

Thus the energy of a configuration is positive and less than $4N$. The next fact shows that an energy drop is irreversible.

**Proposition 1 (Energy is non-increasing).** *Under fully asynchronous dynamics, the energy is a non-increasing function of time and decreases each time a cell with potential $\geq 3$ fires.*

The next theorem shows that configurations of high energy will not appear in the long range behavior of 2D Minority. The next parts are dedicated to the proof of this theorem.

**Main Theorem 1 (Initial energy drop).** *The random variable $T = \min\{t : E(c^t) < 18\lceil \frac{m}{4} \rceil \lceil \frac{n}{4} \rceil\}$ is almost surely finite and $E[T] = O(N^2)$.*

## 3 Proof

The proof is based on the correlation between high energy configurations and local patterns. For example a configuration of energy higher than $3N$ has at least a cell of potential greater than 3. Firing such a cell decreases the energy of the configuration. The bound of $5N/3$ proved in [4] was obtained by considering two facts. Firing two adjacent cells in opposite states of potential 2 decreases the energy of the configuration in two steps. Configurations of energy higher than $5N/3$ admit at least a cell of potential 3 or two adjacent cells in opposite states of potential 2. Here we formalize this approach.

### 3.1 Decreasing sequences

**Definition 5 (decreasing sequence).** *Given a configuration $c$, a finite sequence $\mathcal{S} = (c_i)_{1\leq i\leq j}$ of cells ($j$ is the* length *of the sequence) is a decreasing sequence if firing the $j$ cells $c_1$ to $c_j$ leads to a configuration of lower energy after $j$ steps. The neighborhood $\mathcal{N}(\mathcal{S})$ of a sequence is a set containing the cells of the sequence and their neighbors.*

Note that if for a configuration $c$ a decreasing sequence of length $j$ exists then there exist decreasing sequences of length $k \geq j$. A sequence remains decreasing by adding any cell at the end of the sequence.

**Fact 2 (evolution of a decreasing sequence).** *Given a configuration $c^t$ and a decreasing sequence $\mathcal{S} = (c_i)_{1\leq i\leq j}$ of length $j$. then :*

- *with probability $\frac{1}{N}$ the cell $c_1$ fires : either the energy decreases or $\mathcal{S}' = (c_i)_{2\leq i\leq j}$ is a decreasing sequence of $c^{t+1}$.*
- *with probability $\frac{|\mathcal{N}(\mathcal{S})|-1}{N}$ a cell $c_0$ of $\mathcal{N}(\mathcal{S})$ different from $c_1$ fires : either the potential of $c_0$ is $\geq 3$ and the energy drops or the potential of $c_0$ is 2 and $\mathcal{S}' = (c_i)_{0\leq i\leq j}$ is a decreasing sequence or the potential of $c_0$ is $\leq 1$ and $\mathcal{S}$ is still a decreasing sequence.*
- *with probability $\frac{N-|\mathcal{N}(\mathcal{S})|}{N}$ a cell which is not in $\mathcal{N}(\mathcal{S})$ fires: the energy may drop but $\mathcal{S}$ is still a decreasing sequence.*

**Definition 6 (hypothesis $\mathcal{H}(E,j)$).** *We call $\mathcal{H}(E,j)$ the hypothesis that all configurations of energy $E$ admit a decreasing sequence of size less $j$.*

**Definition 7 (random walks $\mathcal{R}W_j$).** *Given $j \in \mathbb{N}^*$ the random walk $\mathcal{R}W_j$ is a sequence of random variables $(X^t)_{t\geq 0}$ taking their value in $\{0,\ldots,j\}$ such that $X^0 = j$ and :*

| | $P(X^{t+1} = X^t - 1)$ | $P(X^{t+1} = X^t)$ | $P(X^{t+1} = X^t + 1)$ |
|---|---|---|---|
| *if* $X^t = 0$ | *0* | *1* | *0* |
| *if* $X^t \in \{1,\ldots,j-1\}$ | $\frac{1}{N}$ | $\frac{N-5j}{N}$ | $\frac{5j-1}{N}$ |
| *if* $X^t = j$ | $\frac{1}{N}$ | $\frac{N-1}{N}$ | *0* |

**Fact 3.** *Given $j \in \mathbb{N}^*$ and the random walk $\mathcal{R}W_j = (X^t)_{t\geq 0}$ then $T' = \min\{t|X^t = 0\}$ is almost surely finite and $E[T'] = O(j^j N)$.*

**Lemma 1.** *Consider a configuration $c^0$ of energy $E$ and $j \in \mathbb{N}$, suppose that $\mathcal{H}(E,j)$ is true then the random variable $T = \min\{t : E(c^t) < E\}$ is almost surely finite and $E[T] = O(j^j N)$.*

*Proof.* The proof is based on a the coupling between $(c^t)_{t\geq 0}$ and $\mathcal{R}W_j = (X^t)_{t\geq 0}$. To define the coupling we need that at each time step $t$ either $E(c^t) < E$ or $X^t = 0$ or there exists a decreasing sequence $\mathcal{S} = (c_i)_{1\leq i\leq X^t}$ for configuration $c^t$. At each time step $t$ we update $X^t$ and $c^t$ according to the following coupling : if $E(C^t) < E$ or $X^t = 0$ then $X^t$ updates according to the rule of the random walk and independently fire a random cell of $c^t$ chosen uniformly. Otherwise $X^t = k$ with $k \neq 0$ and we consider a decreasing sequence $S^t = (c_i)_{1\leq i\leq k}$ of $c^t$ of size $k$ and :

– if $X^t = j$, with probability $\begin{cases} \frac{1}{N} & X^{t+1} = X^t - 1 \text{ and } c_1 \text{ fires.} \\ \frac{N-1}{N} & X^{t+1} = X^t \text{ and one cell is selected uniformly at random among the } N-1 \text{ other cells.} \end{cases}$

– if $0 < X^t < j$, with prob. $\begin{cases} \frac{1}{N} & X^{t+1} = X^t - 1 \text{ and } c_1 \text{ fires.} \\ \frac{|\mathcal{N}(\mathcal{S})|-1}{N} & X^{t+1} = X^t + 1 \text{ and one cell is selected uniformly at random among } \mathcal{N}(\mathcal{S}). \\ \frac{5j-|\mathcal{N}(\mathcal{S})|}{N} & X^{t+1} = X^t + 1 \text{ and one cell is selected uniformly at random among the } N - |\mathcal{N}(\mathcal{S})| \text{ other cells.} \\ \frac{N-5j}{N} & X^{t+1} = X^t \text{ and one cell is selected uniformly at random among the } N - |\mathcal{N}(\mathcal{S})| \text{ other cells.} \end{cases}$

According to this coupling, each cell of $c^t$ is updated uniformly and $X^t$ evolves according to the rule of the random walk. Now we prove by recurrence over $t$ that this coupling is coherent that is to say that either $E(c^t) < E$ or $X^t = 0$ or there exists a decreasing sequence $\mathcal{S} = (c_i)_{1\leq i\leq X^t}$ for configuration $c^t$. At time $t = 0$, $X^0 = j$ and since $E(c^0) = E$ then by $\mathcal{H}(E, j)$ there exists a decreasing sequence of size $j$. Now if at time $t$ :

– $E(c^t) < E$ then $E(c^{t+1}) < E$ since energy is non-increasing (see Proposition 1).
– $X^t = 0$ then $X^{t+1} = 0$.
– $X^t = j$ and $E(c^t) = E$ then by hypothesis of induction there exists a decreasing sequence $\mathcal{S} = (c_i)_{1\leq i\leq j}$:
If $X^{t+1} = X^t$ then either $E(c^{t+1}) < E$ or $E(c^{t+1}) = E$ and by $\mathcal{H}(E, j)$ there exists a decreasing sequence $\mathcal{S}'$ of size $j$.
If $X^{t+1} = X^t - 1$ then cell $c^0$ fires and according to fact 2 either $E(c^{t+1}) < E$ or $E(c^{t+1}) = E$ and $(c_i)_{2\leq i\leq j}$ is a decreasing sequence.
– $X^t = k$ where $0 < k < j$ and $E(c^t) = E$ then by hypothesis of induction there exists a decreasing sequence $\mathcal{S} = (c_i)_{1\leq i\leq k}$:
If $X^{t+1} = X^t$ then the fired cell is not in $\mathcal{N}(\mathcal{S})$ and either $E(c^{t+1}) < E$ or $E(c^{t+1}) = E$ and $\mathcal{S}$ is still a decreasing sequence.
If $X^{t+1} = X^t - 1$ then cell $c^1$ fires and according to fact 2 either $E(c^{t+1}) < E$ or $E(c^{t+1}) = E$ and $(c_i)_{2\leq i\leq j}$ is a decreasing sequence.
If $X^{t+1} = X^t + 1$ then cell $c^0$ fires and according to fact 2 either $E(c^{t+1}) < E$ or $E(c^{t+1}) = E$ and $(c_i)_{0\leq i\leq j}$ is a decreasing sequence.

Thus the coupling is well defined. We call $T' = \min\{t|X^t = 0\}$. Either $E(c^{T'-1}) < E$ or $E(c^{T'-1}) = E$ and at time $T'-1$ a cell of potential $\geq 3$ (a decreasing sequence of size one) fires. Thus $T \leq T'$ and moreover $E[T] \leq E[T']$. According to lemma 3, $E[T'] = O(j^j N)$ and thus $E[T] = O(j^j N)$.

**Theorem 4.** *If for all configurations of energy $E$ there exists $j \in \mathbb{N}^*$ such that for all $E' > E$ the hypothesis $\mathcal{H}(E', j)$ are true then given a configuration $c^0$ the random variable $T = \min\{t : E(c^t) < E\}$ is almost surely finite and $E[T] = O(j^j N^2)$.*

*Proof.* We call $t_0 = 0$ and for $i \in \mathbb{N}^*$ if $E(c^{t_{i-1}} > E)$ we define $t_i$ as $t_i = \min\{t|E(c^{t_i}) < E(c^{t_{i-1}})\}$. We define $k \in \mathbb{N}$ such that $T = t_k$. According to lemma 1 for $i \leq k$, $E[t_i - t_{i-1}] = O(j^j N)$. And since the energy of a configuration is between 0 and $4N : k < 4N$ and then $E[T] = O(j^j N^2)$

In the next part, we present a method to prove $\mathcal{H}(E, j)$ by considering finite local patterns.

### 3.2 Enumeration of acceptable coloring.

We consider finite patterns of cells and colorings (i.e. the state of the cells) of a pattern. We present an algorithm which enumerates all the possible colorings of a pattern and eliminates colorings which imply a quick energy drop. The remaining colorings are called acceptable. Finally, we compute the maximum energy contained by a pattern matching an acceptable coloring.

**Definition 8 (Pattern).** *A pattern $\mathcal{P}$ is a subset of $\{0, \dots, n\} \times \{0, \dots, m\}$. The pattern $\mathcal{P}$ centered on $c_{ij}$ is the set of cells $\cup_{(k,l)\in\mathcal{P}}\{c_{i+k,j+l}\}$.*

**Definition 9 (Interior).** *The interior of $\mathcal{P}$ is the pattern $\{(i,j) \in \mathcal{P}|(i\pm1, j) \in \mathcal{P}$ and $(i, j\pm1) \in \mathcal{P}\}$.*

**Definition 10 (Coloring).** *A coloring $f$ of $\mathcal{P}$ is a function $f : \mathcal{P} \to Q$. We say that the pattern $\mathcal{P}$ centered on cell $c_{i,j}$ matches coloring $f$ if for any $(k,l) \in \mathcal{P}$, we have $f(k,l) = c_{i+k,j+l}$. We denote by $\mathcal{C}^{\mathcal{P}}$ the set of all the colorings of pattern $\mathcal{P}$.*

**Definition 11 ($k$-acceptable).** *A coloring $f$ of $\mathcal{C}^{\mathcal{P}}$ is $k$-acceptable if there exists a configuration $c$ such that there is no decreasing sequence of length $k$ in $c$ and there is a cell $c_{i,j}$ such that the pattern $\mathcal{P}$ centered on $c_{i,j}$ matches $f$.*

If a coloring $f$ is not $k$-acceptable then $f$ is not $k'$-acceptable for all $k' > k$.

**Definition 12 (Relative Potential).** *Consider a pattern $\mathcal{P}$ centered on cell $c_{ij}$ which matches a coloring $f$ of $\mathcal{C}^{\mathcal{P}}$ and $(k,l) \in \mathcal{P}$, the relative potential $v'_{kl}(f)$ is defined as the number of neighboring cells of cell $c_{i+k,j+l}$ which are in the same state as $c_{i+k,j+l}$ and which are in the pattern $\mathcal{P}$ centered on cell $c_{ij}$.*

**Fact 5.** *Consider a coloring $f$ of $\mathcal{C}^{\mathcal{P}}$, $(k,l) \in \mathcal{P}$ and a pattern $\mathcal{P}$ centered on cell $c_{i,j}$ matching $f$. If $(k,l)$ is in the interior of $\mathcal{P}$ then $v'_{kl}(f) = v_{(k+i)(l+j)}$, otherwise $v'_{kl}(f) \leq v_{(k+i)(l+j)}$.*

A coloring $f$ of $\mathcal{C}^{\mathcal{P}}$ admitting $(i,j) \in \mathcal{P}$ such that $v'_{ij}(f) \geq 3$ is not 1-acceptable.

**Definition 13.** *For each pattern $\mathcal{P}$, we define the oriented graph $G^{\mathcal{P}} = (\mathcal{C}^{\mathcal{P}}, E)$. There is an oriented edge between coloring $f_1$ to coloring $f_2$ if :*

- *there is $(i,j) \in \mathcal{P}$ such that $f_1(i,j) \neq f_2(i,j)$*
- *for all $k,l \neq (i,j)$, $f_1(k,l) = f_2(k,l)$*
- *$v'_{i,j}(f_1) = 2$*

**Fact 6.** *If there is a path of length $i$ from a coloring $f_1$ to a coloring $f_2$ in $G^{\mathcal{P}}$ and if $f_2$ is not $k$-acceptable then $f_1$ is not $(k+i)$-acceptable.*

**Algorithm 1.** *Given a set of coloring $\mathcal{C}_0$, this algorithm computes a set of coloring $\mathcal{C}_1$ such that $\mathcal{C}_1 \subset \mathcal{C}_0$ and for all coloring $f$ of $\mathcal{C}_0 \backslash \mathcal{C}_1$ there exists a path in $G^{\mathcal{P}}$ from $f$ to $f'$ such that $f' \in \mathcal{C}^{\mathcal{P}} \backslash \mathcal{C}_1$ or $f'$ is not 1-acceptable:*

*1 - Compute $\mathcal{C'}_0 = \{f \in \mathcal{C}_0 | \forall (i,j) \in \mathcal{P}, v'_{ij}(f) \leq 2\}$.*

*2 - Compute the graph $G' = (\mathcal{C}'_0 \cup \{d\}, E)$ where there is an edge between two colorings $f$ and $f'$ if and only if there is an edge between $f$ and $f'$ in $G^{\mathcal{P}}$; there is an edge between coloring $f$ and $d$ if there exists a coloring $f' \in \mathcal{C}^{\mathcal{P}} \backslash \mathcal{C}'_0$ such that there is an edge between $f$ and $f'$ in $G^{\mathcal{P}}$.*

*3 - Compute $C_1 = \{f \in \mathcal{C}'_0 |$ there is no path between $f$ and $d$ in $G'\}$.*

The complexity of this algorithm is $O(|\mathcal{C'}_0|^2 + |\mathcal{C}_0|)$.

**Fact 7.** *Consider a set of colorings $\mathcal{C}_0 \subset \mathcal{C}^{\mathcal{P}}$ and $k'$ such that colorings of $\mathcal{C}^{\mathcal{P}} \backslash \mathcal{C}_0$ are not $k'$-acceptable. Consider $\mathcal{C}_1$ the result of algorithm 1 applied to set $\mathcal{C}_0$, then there exists $k$ such that all colorings of $\mathcal{C}^{\mathcal{P}} \backslash \mathcal{C}_1$ are not $k$-acceptable. In particular, this result holds for $\mathcal{C}_0 = \mathcal{C}^{\mathcal{P}}$.*

We apply this algorithm to pattern $\mathcal{P} = \{0, \ldots, 3\}^2$ in order to prove the main theorem. All enumerations of cases in the next proofs are made by computer.

**Theorem 8.** *Consider a configuration $c$. There exists $k$ such that if there exists $(i,j) \in \mathbb{T}$ such that $\sum_{(k,l) \in \{0,\ldots,3\}^2} v_{(i+k)(j+l)} > 18$ then there exists a decreasing sequence of length $k$ in $c$.*

*Proof.* Consider the pattern $\mathcal{P} = \{0, \ldots, 3\}^2$. Applying algorithm 1 to $\mathcal{C}_0 = \mathcal{C}^{\mathcal{P}}$ leads to a set $\mathcal{C}_1$ which contains 1092 colorings whereas $|\mathcal{C}_0| = 65536$. We now consider the pattern $\mathcal{P}' = (\{-1, \ldots, 4\}^2 \backslash \{(-1,-1), (-1,4), (4,-1), (4,4)\})^2$. The interior of $\mathcal{P}'$ is $\mathcal{P}$. Applying algorithm 1 to set $\mathcal{C}'_0 = \{f \in \mathcal{C}^{\mathcal{P}'} | f_{|\mathcal{P}} \in \mathcal{C}_1\}$ leads to a set $\mathcal{C}'_1$. According to fact 7, there exists $k$ such that all colorings of $\mathcal{C}^{\mathcal{P}'} \backslash \mathcal{C}_1$ are not $k$-acceptable. Computing $\mathcal{C}'_1$ using $\mathcal{C}_1$ is much faster than computing $\mathcal{C}'_1$ from $\mathcal{C}^{\mathcal{P}'}$. For each coloring $f$ of $\mathcal{C}_1$, we have $\max_{f' \in \mathcal{C}'_1 | f'_{|\mathcal{P}} = f} (\sum_{(i,j) \in \mathcal{P}} v'_{ij}(f')) \leq 18$. According to fact 5, if the pattern $\mathcal{P}'$ centered on cell $(i,j)$ matches the coloring $f'$ of $\mathcal{C}^{\mathcal{P}'}$ then $\sum_{(k,l) \in \mathcal{P}} v'_{kl}(f') = \sum_{(k,l) \in \mathcal{P}} v_{(i+k)(j+l)}$. Thus if a pattern $\mathcal{P}$ centered on $c_{ij}$ is such that $\sum_{(k,l) \in \mathcal{P}} v_{(i+k)(j+l)} > 18$, then the corresponding coloring is not $k$-acceptable. Thus there is a decreasing sequence of length $k$ in $c$.

**Theorem 9.** *There exists $j$ such that $\mathcal{H}(E,j)$ is true for every $E > 18\lceil\frac{m}{4}\rceil\lceil\frac{n}{4}\rceil$.*

*Proof.* Consider a configuration $c$ and the pattern $\mathcal{P} = \{0,\dots,3\}^2$. There exists $j$ such that if $c_{ij}$ is such that $\sum_{(k,l)\in\mathcal{P}} v_{(i+k)(j+l)} > 18$ then there exists a decreasing sequence of length $j$ in $c$. Suppose that there is no cell $c_{ij}$ such that $\sum_{(k,l)\in\mathcal{P}} v_{(i+k)(j+l)} > 18$. If $n = 0 \mod 4$ and $m = 0 \mod 4$ then $E(c) = \sum_{(i,j)\in\mathcal{T}} v_{ij} = \sum_{0\leq i<n/4, 0\leq j<m/4}(\sum_{(k,l)\in\mathcal{P}}(v_{4i+k,4j+l})) \leq \frac{18nm}{16}$. If $n \neq 0 \mod 4$ and $m = 0 \mod 4$ then $E(c) \leq \sum_{0\leq i\leq n/4, 0\leq j<m/4}(\sum_{(k,l)\in\mathcal{P}}(v_{4i+k,4j+l})) \leq 18\lceil\frac{n}{4}\rceil\frac{m}{4}$. In the two other cases we also have $E(c) \leq \lceil\frac{m}{4}\rceil\lceil\frac{n}{4}\rceil$. This imply that $\mathcal{H}(E,j)$ is true for $E > 18\lceil\frac{m}{4}\rceil\lceil\frac{n}{4}\rceil$.

Applying this result to theorem 4 proves the main theorem.

## References

1. Gács, P.: Reliable computation with cellular automata. Journal of Computer and System Sciences **32**(1) (1986) 15–78
2. Ermentrout, G.B., Edlestein-Keshet, L.: Cellular automata approaches to biological modelling. Journal of Theoretical Biology **160** (1993) 97–133
3. Fatès, N., Morvan, M., Schabanel, N., Thierry, E.: Asynchronous behaviour of double-quiescent elementary cellular automata. In: Proc. of MFCS'2005. Volume 3618 of LNCS., Springer (2005) 316–327
4. Regnault, D., Schabanel, N., Thierry, E.: Progresses in the analysis of stochastic 2d cellular automata: A study of asynchronous 2d minority. In: Proc. of MFCS'2007. Volume LNCS 4708., Springer (2007) 320–332
5. Fatès, N., Regnault, D., Schabanel, N., Thierry, E.: Asynchronous behaviour of double-quiescent elementary cellular automata. In: Proceedings of LATIN'2006. Volume 3887 of LNCS., Springer (2006)
6. Regnault, D., Schabanel, N., Thierry, E.: On the analysis of "simple" 2d stochastic cellular automata. In: Proc. of LATA2008. Volume to appear., Springer (2008)
7. Demongeot, J., Aracena, J., Thuderoz, F., Baum, T.P., Cohen, O.: Genetic regulation networks: circuits, regulons and attractors. C.R. Biologies **326** (2003) 171–188
8. Bersini, H., Detours, V.: Asynchrony induces stability in cellular automata based models. In: Proceedings of Artificial Life IV, Cambridge, MIT Press (1994) 382–387
9. Fatès, N., Morvan, M.: An experimental study of robustness to asynchronism for elementary cellular automata. Complex Systems **16**(1) (2005) 1–27
10. Schönfisch, B., de Roos, A.: Synchronous and asynchronous updating in cellular automata. BioSystems **51** (1999) 123–143
11. Fukś, H.: Probabilistic cellular automata with conserved quantities. Nonlinearity **17**(1) (2004) 159–173
12. Balister, P., Bollobás, B., Kozma, R.: Large deviations for mean fields models of probabilistic cellular automata. Random Structures & Algorithms **29** (2006) 399–415
13. Rojas, R.: Neural Networks: A Systematic Introduction. Springer (1996) Chap. 13 - The Hopfield Model.
14. Goles, E., Martinez, S.: Neural and automata networks, dynamical behavior and applications. Volume 58 of Maths and Applications. Kluwer Academic Publishers (1990)
15. McCoy, B., Wu, T.T.: The Two-Dimensional Ising Model. Harvard University Press (1974)